Markov-type Lie groups in GL(n,R)

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(Received 21 March 1984; accepted for publication 24 August 1984)

The general linear group GL(n,R) is decomposed into a Markov-type Lie group and an abelian scale group. The Markov-type Lie group basis is shown to generate all singly stochastic matrices which are continuously connected to the identity when non-negative parameters are used. A basis is found which shows that it in turn contains a Lie subgroup which corresponds to doubly stochastic matrices, which basis, over the complex field, is shown to give the symmetric group for certain discrete values of the complex parameters. The basis of the Markov algebra is shown to give the negative of the corresponding M-matrices with property "C" (for non-negative combinations). These stochastic Lie groups are shown to be isomorphic to the affine group and the general linear group in one less dimension. The basis generates transformations with a natural interpretation for physical applications.

I. INTRODUCTION

There is extensive literature[1–3] on the general linear group in n dimensions over the real (or complex) field, GL(n,R), which explores various subgroup chains and their representations. Usually these decompositions begin by removing the Lie algebra generator I, leaving the nonsingular unimodular group SL(n,R). Further restrictions requiring the invariance of some bilinear form leads to subsequent decomposition and in particular the determination of all simple Lie algebras. This paper will explore an alternative decomposition of GL(n,R) requiring the invariance of a linear form and resulting in a solvable (not semisimple) Lie group chain with Markov-type Lie groups and their associated Lie algebras down to the symmetric group. Butler and King[4] have extensively explored the symmetric group as a subgroup of the general linear group and have introduced two ideas which we explore more fully: (1) the invariance of a linear form in GL(n,R) and (2) the concept of the symmetric group S_n as a subgroup of GL(n,R).

Requiring the invariance of a linear form

$$\sum x_i$$

is closely related to singly (and doubly) stochastic processes which leave Σx_i invariant and $x_i \geqslant 0$. First studied by Markov[5] in 1907, a singly (row) stochastic or Markov process is a linear transformation $M_{ij} \geqslant 0$ with

$$\sum_i M_{ij} = 1, \tag{1.1}$$

which can be thought of as transforming a vector of probabilities (or occupation numbers) $x_i \geqslant 0$ into a new set $x_i' = M_{ij} x_j$ and is also doubly stochastic if

$$\sum_j M_{ij} = 1. \tag{1.2}$$

Markov processes only form a semigroup since, in general, they do not process an inverse.[6]

In Sec. II we will study the decomposition of GL($2,R$) into a Markov-type Lie group and an abelian scale group. Specifically it will be shown that all Markov processes continuously connected to the identity are all generated by a certain basis for its Lie algebra with non-negative linear combinations. In Sec. III we will extend these ideas to n dimensions and discuss a connection to S_n illustrating that the permutations are Markov processes which can be reached from the identity with the same Lie algebra over the complex field.

In Sec. IV we briefly discuss the invariance of indefinite linear forms $\Sigma x_i - \Sigma y_j$. Section V is a general discussion of properties of the Markov Lie group. In particular it is shown that all analytic functions of the basis are linear and thus no Casimir operators exist. In Sec. VI a basis for a doubly stochastic Lie algebra is obtained and related to the symmetric group in Sec. VII. A close connection between the Markov Lie algebra and the M-matrices with property "C" is established in Sec. VIII with general conclusions following in Sec. IX.

II. NOTATION AND DEFINITION OF $M(n,R)$ IN TWO DIMENSIONS

We define the "Markov" Lie group $M(n,R)$ to be the subgroup of GL(n,R) which preserves

$$\sum x_i,$$

where x_i are the vector components $i = 1 \cdots n$ acted upon by the $n \times n$ representation of GL(n,R). We define the vectors $\langle 1|$ and $|1\rangle$ to be row and column vectors, respectively, with all components equal to 1. It follows that $\langle 1|M|x\rangle = \langle 1|x\rangle$ is equivalent to

$$\sum_i M_{ij} = 1, \tag{2.1}$$

for all j. This is equivalent to the preservation of a linear rather than a bilinear form. The subset consisting of all $M_{ij} \geqslant 0$ would not be useful unless the M_{ij} are smoothly connected in the group space and have a useful form as we now show.

The infinitesimal transformation which takes a positive fraction $0 \leqslant \epsilon \leqslant 1$ of a component and adds it to the other component will preserve the sum and will always be positive when acting upon non-negative components. It also has the natural interpretation of a transition probability for a time ϵ. It can be written in two dimensions as

 0022-2488/85/020252-06$02.50

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = 1 + \epsilon \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = 1 + \epsilon \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.2}$$

transferring the fraction ϵx_2 to x_1 and ϵx_1 to x_2, respectively. Defining $M(2,R)$ in terms of the basis

$$L^{12} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

and

$$L^{21} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

one verifies that

$$e^{\lambda L^{12}} = \begin{pmatrix} 1 & 1-e^{-\lambda} \\ 0 & e^{-\lambda} \end{pmatrix}$$

and

$$e^{\lambda L^{21}} = \begin{pmatrix} e^{-\lambda} & 0 \\ 1-e^{-\lambda} & 1 \end{pmatrix} \tag{2.4}$$

and that $\langle 1 | e^{\lambda L^{12}} = \langle 1 | e^{\lambda L^{12}} = \langle 1 |$ as required (λ real). (2.5)

One also verifies that $[L^{12}, L^{21}] = +L^{12} - L^{21}$, giving the structure constants. Closure of the group can be seen from closure of the L^{12} and L^{21} commutation rules or from sequences of infinitesimal transformations which individually and thus collectively preserve $\langle 1 | x \rangle$. Thus in two dimensions the most general form of $M(2,R)$ is

$$e^{\lambda \cdot L} = e^{\lambda_{12} L^{12} + \lambda_{21} L^{21}} \tag{2.6}$$

with the group inverse $e^{-\lambda \cdot L}$ and group unit with $\lambda_{ij} = 0$. GL(n,R) itself has the additional basis elements

$$L^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$L^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

no combination of which preserves $\langle 1 | x \rangle$. The Lie group $M(2,R)$ thus satisfies the requirement of preserving the linear form $\langle 1 | x \rangle$, but as λ ranges over the reals there is an unphysical region when either $\lambda_{i \neq j} < 0$, which will not give a Markov matrix, as well as a physical region with both $\lambda_{i \neq j} \geqslant 0$, which always gives an acceptable Markov matrix. Like GL(n,R), $M(n,R)$ is noncompact. The limit points at $\lambda = \infty$ give the singular transformations $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ with λ_{12} and λ_{21}, respectively.

III. GENERALIZATION TO n DIMENSIONS

These results are easily generalized to n dimensions where we define

$$L^{ij}_{kl} \equiv \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{jl} \tag{3.1}$$

for $i \neq j$ to be the kl element of the L^{ij} linear operator. Similarly $(L^{ii})_{kl} \equiv \delta_{ik}\delta_{il}$. The $(n^2 - n) L^{ij}$ matrices and the (n) L^{ii} matrices form a basis for the Lie algebra which generates GL(n,R). This can be seen by forming the n^2 combinations which possess a 1 at only one position in the matrix with zeros elsewhere. We define $M(n,R)$ and $A(n,R)$ to be the matrices generated by the L^{ij} and the L^{ii}, respectively. Thus $\mathrm{GL}(n,R) = A(n,R) \oplus M(n,R)$ for their respective Lie algebras.

That the L^{ii} generate an abelian subgroup of order n, $A(n,R)$, of GL(n,R) follows immediately from the general form

$$e^{\Sigma \lambda_{ii} L^{ii}} = \begin{pmatrix} e^{\lambda_{11}} & & & \\ & e^{\lambda_{22}} & & \\ & & \vdots & \\ & & & e^{\lambda_{nn}} \end{pmatrix} \tag{3.2}$$

which scales the ith coordinate by $e^{\lambda_{ii}}$. It is closed, noncompact, has the inverse

$$e^{-\lambda_{ii} L^{ii}} \tag{3.3}$$

and a unit defined by $\lambda_{ii} = 0$. A unimodular subalgebra is obtained by redefining the basis as

$$I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}, \quad H_4 = \cdots, \tag{3.4}$$

with H_i as a diagonal traceless basis with $i = 2,3,\ldots,n$.

The L^{ij} $(i \neq j)$ in three dimensions take the form

$$L^{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L^{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$
$$L^{31} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L^{21} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$
$$L^{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad L^{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which follows from writing the infinitesimal transformation which subtracts ϵx_j from the jth component and adds ϵx_j to the ith component. Thus these infinitesimal transformations preserve $\langle 1 | x \rangle$ individually and collectively and thus any group element

$$e^{\lambda_{ij} L^{ij}}$$

compounded from sequences of infinitesimal transformations also preserves $\langle 1 | x \rangle$. Conversely all linear transformations in GL(n,R) which preserve $\langle 1 | x \rangle$ are included in the basis since $\langle 1 | (1 + \Sigma \epsilon_{ij} L^{ij}) = \langle 1 |$ implies that $\Sigma \epsilon_{ij} L^{ij} = 0$ over a column and the $n - 1$ different linear combinations using L^{ij} for a fixed j spans all such possible combinations. Consequently $M(n,R)$ contains all those and only those transformations in GL(n,R) which preserve $\langle 1 | x \rangle$. Furthermore it is both necessary and sufficient that $\lambda_{ij} \geqslant 0$ for all i and j in order to guarantee that any vector with all nonnegative components is transformed into a vector with nonnegative components. This can be seen by looking at the

most general infinitesimal transformation which is seen to be non-negative and thus all products of these are also. Thus for real λ_{ij}, all Markov transformations in GL(n,R) continuously connected to the identity are those elements in $M(n,R)$ formed with $\lambda_{ij} \geqslant 0$. The closure of $M(n,R)$ can be shown from the closure of the commutators of the generating Lie algebra

$$\sum_i \sum_j (L_{ij}^{lm} L_{jk}^{rs} - \Sigma L_{ij}^{rs} L_{jk}^{lm}) = 0, \tag{3.6}$$

which demonstrates that the commutator must be a combination of matrices with a zero row sum for each column. Thus the commutator is a linear combination of elements of the algebra. Also the product of two elements of $M(n,R)$ (with unit row sums) is

$$\sum_i \sum_j M_{ij}^a M_{jk}^b = \sum_j M_{jk} = 1 \tag{3.7}$$

and thus is a member of $M(n,R)$. The unit operator is produced with $\lambda_{ij} = 0$ and the inverse with $-\lambda_{ij}$. Thus $M(n,R)$ is a Lie group with L^{ij} $(i \neq j)$ forming the basis of its Lie algebra. (Antisymmetry and the Jacobi identity follow automatically from a matrix definition.)

Although we found all Markov matrices in GL(n,R) with real λ_{ij}, one can ask if there are acceptable real Markov matrices arising from complex λ_{ij}. It is easy to verify that none are in the neighborhood of the identity. However consider

$$e^{\lambda(L^{12} + L^{21})} = \frac{1}{2} \begin{pmatrix} 1 + e^{-2\lambda} & 1 - e^{-2\lambda} \\ 1 - e^{-2\lambda} & 1 + e^{-2\lambda} \end{pmatrix}, \tag{3.8}$$

for imaginary λ, which give real matrices. One can obtain $e^{-2\lambda} = -1$ with $-2\lambda = \pm i n_0 \pi$ or

$$\lambda = n_0 i\pi/2, \tag{3.9}$$

where n_0 is an odd integer. This gives

$$e^{\lambda L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ for } n_0 = 1$$

which is a permutation (transposition) of the two variables. Thus using these discrete imaginary values for λ with $(L^{ij} + L^{ji})$ one obtains the transpositions between any two pairs of variables and, by multiplication of these, any permutation. Thus the permutation (symmetric) group is contained in $M(n,C)$ for certain discrete complex values of the group parameters (that a transposition is continuously connected to the identity only with complex parameters, is easily proven by diagonalizing the transposition matrix).

IV. TRANSFORMATIONS PRESERVING $\Sigma x_i - \Sigma y_i$

Beginning with an example in two dimensions, we can ask for transformations in GL(n,R) which preserve $x - y$. The above results on the Markov matrices suggest infinitesimal transformations which add or subtract a fraction of either coordinate to the other. Thus we define

$$\begin{aligned} L^{-12} &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = L^{12} + 2L^{22}, \\ L^{-21} &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = L^{21} + 2L^{11}, \end{aligned} \tag{4.1}$$

which give

$$\begin{aligned} e^{\lambda L^{-12}} &= \begin{pmatrix} 1 & -1 + e^{+\lambda} \\ 0 & e^{+\lambda} \end{pmatrix}, \\ e^{\lambda L^{-21}} &= \begin{pmatrix} e^{+\lambda} & 0 \\ -1 + e^{+\lambda} & 1 \end{pmatrix}, \end{aligned} \tag{4.2}$$

respectively. In n dimensions these matrices give the correct prescription for the connection between the positive definite and negative definite subspaces. The invariant form can be written as $\langle 1|\eta|x\rangle$ where η is a metric which carries the sign for the negative definite portions of the space. We will refer to these transformations as indefinite Markov transformations $M(r+s,R)$, where r and s are the dimensions of the positive definite and negative definite subspaces. The $M(r+s,R)$ transformations also form a Lie group and give physically acceptable vectors $(x_i \geqslant 0)$ when any element $\lambda_{ij} \geqslant 0$ acts upon a physically acceptable vector.

V. GENERAL PROPERTIES OF $M(n,R)$

Geometrically, $M(n,R)$ can be viewed as giving all nonsingular linear transformations on the hyperplane perpendicular to the vector $\langle 1| = (1,1,1,...,1)$ since $\langle 1|M = \langle 1|$ or equivalently since $\Sigma x_i = \text{const}$ is the equation for the hyperplane and invariant. For non-negative λ_{ij}, $e^{\lambda \cdot L}$ maps the positive quadrant into itself. In fact, from an arbitrary point $x_i \geqslant 0$ any other point $x_i \geqslant 0$ can be reached with $M(n,R)$. A particular λ_{ij} determines the fraction of the jth sector which is added to the ith sector. If y_i are defined by $y_i^2 = x_i$ then $M(n,R)$ maps the sphere $\Sigma y_i^2 = \text{const}$ into itself for $\lambda_{ij} \geqslant 0$ and thus behaves like a nonlinear representation of the rotation group but without an inverse. Likewise in two dimensions, $M(1+1,R)$ preserves $y_0^2 - y_1^2$ and thus behaves like a nonlinear representation of the Lorentz group. The invariant hyperplane of $M(r+s,R)$ is

$$\sum_{i=1}^{r} x_i - \sum_{i=r+1}^{r+s} x_i = \text{const}. \tag{5.1}$$

All of the physical portion of the space can be covered with $M(r+s,R)$ from the initial state with $x_i^{\max} = c$, $x_{j\neq i} = 0$.

The group $M(n,R)$ is not unimodular (determinant $\neq 1$) since the basis of its algebra, L^{ij}, is not traceless. Consequently $M(n,R)$ is not contained in SL(n,C). By evaluating the Killing form, $g_{ij} = c_{ik}^{1} C_{jl}^{k}$ in two dimensions one obtains

$$g = |g_{ih}| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0. \tag{5.2}$$

Since a Lie algebra is semisimple if and only if $g \neq 0$, it follows that $M(2,R)$ is not semisimple. Defining $L = L^{12} - L^{21}$, one can show $[L,L^{12}] = L = [L,L^{21}]$ and thus L forms an invariant subalgebra or ideal. Consequently S is not simple. $M(n,R)$ is also noncompact since the parameter space is unbounded.

Generally one can prove that $M(n,R)$ is isomorphic to the affine group in $n-1$ dimensions (consisting of the general linear group and translations). This follows from the result above that $M(n,R)$ consists of linear transformations in GL(n,R) which are restricted to transformations in the hyperplane perpendicular to $|1\rangle$, which is a space of dimension $(n-1)$. The actual isomorphism can be implemented by a

coordinate transformation, R, which rotates the x_n axis into the vector $|1\rangle$ after which all of the $n^2 - n$ linear transformations which were previously in the hyperplane now become linear transformations on the subspace $x_1 x_2 \cdots x_{n-1}$, leaving the x_n axis invariant. The transformed $M(n,R)$ matrices then take the customary form for the affine group:

$$\begin{pmatrix} \mathrm{GL}(n-1,R) & T(n-1) \\ 0 & 1 \end{pmatrix}.$$

Thus all properties and representations of $M(n,R)$ are those of the affine group in $(n-1)$ dimensions.

For semisimple Lie groups, the irreducible representations are classified by the spectra of Casimir operators[7]

$$I_n = C^{\beta_2}_{\alpha_1\beta_1} C^{\beta_3}_{\alpha_2\beta_2} \cdots C^{\beta_1}_{\alpha_n\beta_n} L^{\alpha_1} L^{\alpha_2} \cdots L^{\alpha_3}, \tag{5.3}$$

which commute with all the elements of the algebra. Normally I_n is defined only for semisimple algebras but an interesting nonexistence proof is possible for $M(n,R)$ for representations of the form (3.1): For two elements L^a and L^b, in a representation of arbitrary order, we have

$$\sum_i \sum_j L^a_{ij} L^b_{jk} = \sum_j \sum_i L^a_{ij} L^b_{jk} = 0, \tag{5.4}$$

showing that the product of two matrices with

$$\sum_i L_{ij} = 0 \tag{5.5}$$

is again a matrix of this type. But since the L^{ij} are a complete basis of all such matrices it follows that any product is expressible as a linear combination:

$$L^{ij} L^{rs} = \Sigma \lambda_{lm} L^{lm}. \tag{5.6}$$

Consequently any analytic function of the L^{ij} is expressible as a linear combination of the L^{ij} and thus no operator like the Casimir operators exist for $M(n,R)$ for representations of the form (3.1). The generality of this proof rests upon the fact that the L generate an algebra of arbitrary order n. In fact the general group element

$$M = e^{\lambda \cdot L} = 1 + \lambda \cdot L + (1/2!)(\lambda \cdot L)^2 + \cdots \tag{5.7}$$

must therefore be repressible as $M = 1 + a_{ij}(\lambda) L^{ij}$ where the a_{ij} are functions of the λ_{ij} and must all satisfy $0 \leqslant a_{ij} \leqslant 1$.

It would be important to have a useful expression for the functions $a_{ij}(\lambda)$ as well as for the inverse functions because the $a_{ij}(\lambda)$ give the detailed connection between any particular Markov transformation and the element of the Lie algebra which generates it. In this paper we have only established existence and general properties of this connection.

VI. THE DOUBLY STOCHASTIC SUBGROUP

In certain applications of Markov or stochastic processes an additional requirement, $M|1\rangle = 1$, is imposed (in addition to $\langle 1|M = \langle 1|$). These transformations are termed doubly stochastic and have both unit row and unit column sums. We denote the collection of real nonsingular doubly stochastic transformations on an n-dimensional space as $M^D(n,R)$. By considering the infinitesimal transformations

$$M^D = 1 + \epsilon_{ij} L^{D_{ij}}, \tag{6.1}$$

it follows that it is necessary and sufficient that

$$\sum_m L^{D_{ij}}_{lm} = 0. \tag{6.2}$$

It can be seen that this imposes $n-1$ independent conditions on the L^{ij} since the nth row sum will follow from the zero column sums. That

$$L^{D_{ij}}$$

forms a Lie algebra follows from

$$\sum_r L^{D_{ij}}_{mn} L^{D_{kl}}_{nr} = 0, \tag{6.3}$$

thus the product of two elements must be a linear combination of a complete basis of L^D. That result is stronger than necessary for the commutator to be expressible in terms of the basis elements. As a consequence of the expression of the product as a member of the algebra it follows, as for singly stochastic processes, that any analytic function of

$$L^{D_{ij}}$$

is linearly expressible in terms of the L^D basis and thus is a member of the algebra. It also follows that

$$M^D = e^{\lambda \cdot L^D} = +\lambda L^D + \cdots = 1 + \alpha \cdot L^D, \tag{6.4}$$

where α is the linear combination is detemined by the λ. The proof follows from the products being expressible as elements of the algebra which gives a linear combination of elements which is an element of the algebra

$$\alpha \cdot L = \alpha_{ij} L^{D_{ij}}. \tag{6.5}$$

(Convergence is guaranteed for the exponential.) A basis for the Lie algebra L^D can be constructed by taking certain combinations of the L^{ij} generators which give vanishing row sums. The $(n^2 - n) L^{ij}$ must satisfy $n-1$ independent restrictions giving $(n-1)^2$ independent

$$L^{D_{ij}}.$$

We will absorb the $n-1$ constraints by using the $n-1$ elements on the diagonal just below the main diagonal. We define

$$L^{D_{ij}},$$

beginning with L^{ij}:

$$\begin{pmatrix} 0 & & & & & & \\ & 0 & & & 1 & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & -1 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}, \tag{6.6}$$

where one observes that the row sums can always be made zero by adding the terms

$$L^{j,j-1} + L^{j-1,j-2} + \cdots + L^{i+1,i}, \tag{6.7}$$

which takes the form

$$\begin{pmatrix} 0 & & & & & & \\ & -1 & \cdots & & +1 & & \\ & 1 & -1 & & & & \\ & & 1 & -1 & & & \\ & & & 1 & -1 & & \\ & & & & 0 & 0 & \\ & & & & & & 0 \end{pmatrix} \tag{6.8}$$

$(+1$ in i,j position) (case for $i<j$).

If $i>j$ then the sequence is

$$L^{D_{ij}} = L^{ij} + L^{j,j+1} + L^{j-1,j-2} \ldots L^{2,1} + L^{i+1,j} + L^{i+2,i+1} \ldots L^{n,n-1} + L^{n,n}, \quad (6.9)$$

which takes the form

$$\begin{pmatrix} -1 & & \cdots & & & 1 \\ 1 & -1 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & +1 & & & & 1 \\ & & & & 1 & -1 \end{pmatrix} \quad (6.10)$$

The basis for the Markov (singly stochastic process) could be taken as the

$$(n-1)^2 L^{D_{ij}},$$

along with the $(n-1) L^{i,i-1}$.

The Lie group $M^D(n,R)$ can be proved to be isomorphic to $GL(n,R)$ by referring to the rotation R which transformed the x_n axis into the vector $|1\rangle$ in Sec. III. That transformation R showed that $M(n,R)$ was isomorphic to the affine group which contains the $(n-1)$-dimensional translation group on the remaining $x_1 x_2 \cdots x_{n-1}$ coordinates. A restriction of $M(n,R)$ to $M^D(n,R)$ imposes the requirement that the vector $|1\rangle$ is invariant ($n-1$ new constraints) and thus in the R transformed coordinates the origin must be invariant. The origin is left invariant by disallowing the translation portion of the affine group in $(n-1)$ dimensions, thus leaving the allowable transformations as $GL(n-1,R)$ which is thus isomorphic to $M^D(n,R)$.

VII. CONNECTION TO S_n

The symmetric (permutation) group S_n is nonsingular and thus is in $GL(n,C)$ for certain values of the λ's in the generating Lie algebra. Furthermore, since S_n must permute each element into some new position, it must consist of exactly a single one in each row and each column (giving $n!$ possible matrices). Thus S_n must not only be Markov [in $M(n,C)$]; it must also be doubly stochastic [thus in $M^D(n,C)$]. Thus the $n!$ elements of S_n must be generated by some set of λ_{ij} in the Lie algebra $M^D(n,C)$. As $n! > (n-1)^2$ for all n it follows that some of the

$$L^{D_{ij}}$$

must generate several of the S_n elements. Furthermore, if

$$e^{\lambda \cdot L^D} \in S_n,$$

then, because of closure of S_n,

$$e^{m\lambda \cdot L^D} \in S_n, \quad (7.1)$$

for all integers m. Using the previous result that

$$e^{\lambda \cdot L^D} = 1 + \alpha \cdot L^D \quad (7.2)$$

and that S_n must be contained in

$$e^{\lambda \cdot L^D},$$

then it follows that S_n must be contained in

$$1 + \alpha \cdot L^D,$$

for selected values of α_{ij}. In particular when a single $\alpha_{ij} = 1$, others $= 0$, one obtains the permutations

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 0 & & & & \\ & & 1 & 0 & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & & & & & & 1 \\ 1 & 0 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & 1 & & & & & 0 \end{pmatrix}. \quad (7.3)$$

Thus

$$L^{D_{ij}}(i<j)$$

gives the permutation $x_1 x_2 (x_i \cdots x_j) \cdots x_n$ and

$$L^{D_{ij}}(i>j+1)$$

gives the permutation

$$x_1 x_2 \cdots x_j) x_{j+1} \cdots (x_i \cdots,$$

where terms outside the parentheses are unchanged and those inside are cyclically permuted to the right. The fundamental permutations can be simply represented by ordered pairs (i,j), which are defined $i,j = 1,\cdots n$ with $i \neq j$ and $i \neq j-1$. They are fundamental in the sense that there is a one-to-one correspondence between these $(n-1)^2$ permutations and the doubly stochastic Lie algebra basis which contains S_n.

VIII. CONNECTION TO *M*-MATRICES

M-matrices form an important class of matrices which are connected to the theory of Markov matrices. An *M*-matrix A can be defined by $A = sI - B$, where $s > 0$, $B_{ij} > 0$ and where $s \geqslant \rho(b)$ is the spectral radius of B. The form of A_{ij} is

$$\begin{pmatrix} a_{11} & -a_{12} & a_{13} & \\ -a_{21} & a_{22} & -a_{23} & \vdots \\ & \cdots & & \end{pmatrix},$$

with $a_{ij} \geqslant 0$ (non-negative diagonal and nonpositive off diagonal terms). Extensive literature has developed relating *M*-matrices to Markov matrices and to non-negative matrices in general. In particular it can be shown that if B is a Markov matrix then $A = 1 - B$ is an *M*-matrix with "property C" (rank A = rank A^2).

We have previously proved that a Markov matrix $B = e^{\lambda \cdot L}(\lambda_{ij} \geqslant 0)$ has the representation $B = 1 + \alpha \cdot L$, where the $\alpha_{ij} \geqslant 0$ are determined by the λ_{ij}. Thus it follows from $-\alpha \cdot L = 1 - B$ that $-\alpha \cdot L$ is an *M*-matrix with property *C*

$(\alpha_{ij} \geqslant 0)$. Thus all those elements of the Markov Lie algebra, which are acceptable generators of Markov transformations, are the negative of an M-matrix with property C.

IX. CONCLUSIONS

We have studied a decomposition of the general linear group $GL(n,R) = A(n,R) \oplus M(n,R)$, where $A(n,R)$ is the abelian scale transformation in n dimensions which naturally separates into the unit I and the $(n-1)H_i$ traceless generators. $M(n,R)$ was defined by $\langle 1|M = \langle 1|$, preserving

$$\sum x_i,$$

and was shown to give all Markov matrices continuously connected to the identity when the parameters in the associated Lie algebra were non-negative. Thus, even though Markov transformations do not form a group, they can be studied using much of the power and theorems available with Lie algebras. $M(n,R)$ was shown to contain a subgroup $M^D(n,R)$ of doubly stochastic processes and a basis of the $(n-1)^2$ generators of its Lie algebra were found. The M^D subalgebra was shown to contain the discrete symmetric group on n symbols, S_n, for certain values of the parameters over the complex field for which the transformations become real. Likewise the abelian group over the complex field $A(n,C)$ contains the real inversions. Thus the real transformations in $GL(n,C)$ consist of those continuously connected to the identity through real parameters and the "discrete" groups which consist of those real transformations (inversions and the symmetric group) which can only be reached from the identity with complex parameters. Thus one can ask what restrictions are placed on behavior of representations of real Lie groups under the associated discrete groups which can be reached through complex parameters.

All subgroups of $GL(n,C)$ can be viewed as a simultaneous implementation of

$$I, H_i, L^{D_{ij}}$$

and the $L^{i+1\,i}$ and thus as simultaneous scaling-, Markov-, and doubly stochastic-type transformations. In particular, the importance of classifying tensors under S_n can be seen here from a different point of view.

The Lie group approach to Markov processes allows one to formally use some alternative approaches: If the actual Markov transformation is uncertain but one knows the probability that a given transformation is correct then the transformation can be written

$$\int \eta(\lambda_{ij}) e^{\lambda_{ij} L^{ij}}\, d\lambda_{ij}, \tag{9.1}$$

where η represents a statistical weighting for different transformations. Since $e^{\lambda \cdot L} = 1 + \alpha \cdot L$ and since one requires

$$\int \eta\, d\lambda = 1, \tag{9.2}$$

then it follows that there exists a β such that

$$\int \eta(\lambda) e^{\lambda \cdot L}\, d\lambda = e^{\beta \cdot L}, \tag{9.3}$$

showing that statistical weightings of Markov processes are a single Markov process.

ACKNOWLEDGMENT

The author is indebted to the referee for a number of very useful comments and criticisms.

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